

# Frobenius Structural Matrix Algebras

S. Dăscălescu<sup>1</sup>, M. C. Iovanov<sup>1,2</sup> and S. Predut<sup>1</sup>

<sup>1</sup>University of Bucharest, Facultatea de Matematică,  
Str. Academiei 14, Bucharest 1, RO-010014, Romania,

<sup>2</sup> University of Iowa, 14 MacLean Hall, Iowa City, Iowa 52242, USA

e-mail: sdascal@fmi.unibuc.ro, miodrag-iovanov@uiowa.edu, sina.predut@gmail.com

## Abstract

We discuss when the incidence coalgebra of a locally finite preordered set is right co-Frobenius. As a consequence, we obtain that a structural matrix algebra over a field  $k$  is Frobenius if and only if it consists, up to a permutation of rows and columns, of diagonal blocks which are full matrix algebras over  $k$ .

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## 0 Introduction and preliminaries

Let  $k$  be a field. A structural matrix algebra is an algebra consisting of all matrices in some  $M_n(k)$  which have zeros on certain prescribed positions. This kind of algebras are present in ring theory by the large number of examples and counterexamples using them. Also, certain structural matrix algebras are important in the study of numerical invariants of PI algebras. Structural matrix algebras have been studied in for example [19], [4], [1].

On the other hand, Frobenius algebras have been a theme of research with applications in representation theory, topology, quantum Yang-Baxter equation, conformal field theory, Hopf algebra theory; see [14], [15], [18]. It is a natural question to ask when is a structural matrix algebra Frobenius. We will answer this question in the present note. In fact we will investigate a more general situation. We consider a locally finite preordered set  $(X, \leq)$ , i.e. the relation  $\leq$  is reflexive and transitive, and the interval  $[x, y] = \{z \mid x \leq z \leq y\}$  is finite for any  $x \leq y$ . We consider the vector space  $C = IC(X)$  with basis a set  $\{e_{x,y} \mid x, y \in X, x \leq y\}$ . This space has a coalgebra structure with comultiplication  $\Delta$  and counit  $\varepsilon$  defined on basis elements by

$$\Delta(e_{x,y}) = \sum_{x \leq z \leq y} e_{x,z} \otimes e_{z,y}$$

$$\varepsilon(e_{x,y}) = \delta_{x,y}$$

for any  $x, y \in X$  with  $x \leq y$ ; here  $\delta_{x,y}$  denotes Kronecker's delta. Incidence coalgebras have been of a great interest for combinatorics, see for instance [13], [17]. We will discuss when is this incidence

coalgebra right co-Frobenius, i.e. when  $C$  embeds in  $C^*$  as a right  $C^*$ -module. Co-Frobenius coalgebras have a rich representation theory, also their study is interesting since a Hopf algebra has non-zero integrals if and only if its underlying coalgebra is right co-Frobenius. A similar problem was posed in [8] in the particular case of incidence coalgebras of locally finite ordered sets. In that paper, an equivalent characterization of right co-Frobenius coalgebras is considered, more precisely the existence of a right non-singular  $C^*$ -balanced bilinear form on  $C$ , and the method is to compute all such bilinear forms. We propose here a completely different approach, by using corepresentation (i.e. comodule) theory of  $C$ . Since a finite dimensional algebra is Frobenius if and only if its dual coalgebra is right (or left) co-Frobenius, and the structural matrix algebras are precisely the incidence algebras of finite preordered sets (so they are isomorphic to the dual algebras of incidence coalgebras of finite preordered sets), we obtain as an application the structure of all Frobenius structural matrix algebras. We refer to [7] for basic facts about coalgebras and comodules. We will freely regard a right  $C$ -comodule as a left  $C^*$ -module in the usual way.

## 1 The coradical filtration

We start with a general remark on coalgebras having a special kind of basis. We will then apply it to incidence coalgebras. Let  $C$  be an arbitrary coalgebra with comultiplication  $\Delta$ . The coradical  $C_0$  of  $C$  is the sum of all simple subcoalgebras of  $C$ , and this coincides with the socle of  $C$  as a right (or as a left)  $C$ -comodule. If  $U$  and  $V$  are subspaces of  $C$ , the wedge  $U \wedge V$  is defined by  $U \wedge V = \Delta^{-1}(U \otimes C + C \otimes V)$ . The coradical filtration  $C_0 \subseteq C_1 \subseteq \dots \subseteq C_n \subseteq \dots$  of  $C$  is defined by  $C_n = C_0 \wedge C_{n-1}$  for any  $n \geq 1$ .

**Definition 1.1** (i) We say that  $\mathcal{B}$  is a coradical basis of  $C$  if  $\mathcal{B}$  is a  $k$ -basis of  $C$  and  $\mathcal{B} \cap C_n$  is a basis of  $C_n$  for all  $n$ .

(ii) A right (or left) coideal  $M$  of  $C$  will be called  $\mathcal{B}$ -supported if  $\mathcal{B} \cap M$  is a  $k$ -basis of  $M$ .

We say that a non-zero (co)module is local if it has a unique maximal sub-(co)module. The following result is obvious.

**Proposition 1.2** Let  $C$  be a coalgebra with a coradical basis  $\mathcal{B}$  and let  $M$  be a finite dimensional local right coideal of  $C$  which is  $\mathcal{B}$ -supported. Then  $M$  is generated as a right  $C$ -comodule by any element in the nonempty set  $(\mathcal{B} \cap M) \setminus \text{Jac}(M)$  (here,  $\text{Jac}(M)$  denotes the Jacobson radical of  $M$ ).

Let now  $C$  be the incidence coalgebra  $IC(X)$  of a locally finite preordered set  $X$ .

Let  $\sim$  be the equivalence relation on  $X$  defined by  $x \sim y$  if and only if  $x \leq y$  and  $y \leq x$ . We denote by  $(\mathcal{C}_i)_{i \in I}$  the set of equivalence classes with respect to  $\sim$ , which are all finite since  $\leq$  is locally finite. Then for any  $i \in I$ , the space  $D_i = \sum_{x,y \in \mathcal{C}_i} ke_{x,y}$  is a subcoalgebra of  $C$  isomorphic to the matrix coalgebra  $M^c(n_i, k)$ , the dual coalgebra of the matrix algebra  $M_{n_i}(k)$ . In particular  $D_i$  is a simple coalgebra. If  $i \in I$  and  $x \in \mathcal{C}_i$ , define  $S_x = \sum_{y \in \mathcal{C}_i} ke_{x,y}$ . Then  $S_x$  is a simple right  $C$ -comodule and  $D_i = \sum_{x \in \mathcal{C}_i} S_x$ .

For any  $x \in X$  we also consider the space  $E_x = \langle e_{x,y} \mid y \in X, x \leq y \rangle$ , which is a right  $C$ -subcomodule of  $C$ . Obviously  $S_x \subseteq E_x$  and  $C = \bigoplus_{x \in X} E_x$ , thus any  $E_x$  is an injective right  $C$ -comodule.

If  $x \leq y$ , we say that the length of the interval  $[x, y]$  is 0 if  $x \sim y$ , and the length of  $[x, y]$  is  $n > 0$  if  $n$  is the greatest possible such that there exists a sequence  $x = x_0 < x_1 < \dots < x_n = y$  in  $X$ .

**Proposition 1.3** (i) *The coradical  $C_0$  of  $C$  is  $\sum_{i \in I} D_i$ .*

(ii)  *$C_n$ , the  $(n+1)$ th term of the coradical filtration of  $C$ , is the subspace spanned by all  $e_{x,y}$  such that  $[x, y]$  has length at most  $n$ .*

**Proof:** Denote  $D = \sum_{i \in I} D_i$ . Since  $D$  is a sum of simple subcoalgebras, we see that  $D \subseteq C_0$ .

It is straightforward to show by induction on  $n$  that  $\wedge^{n+1} D$  is the subspace spanned by all  $e_{x,y}$  such that  $[x, y]$  has length at most  $n$ . It follows that  $\bigcup_n (\wedge^{n+1} D) = C$ , where  $\wedge^{n+1} D$  is the space obtained by wedging  $n$  times  $D$  with itself. Then  $C_0 \subseteq D$  by [7, Exercise 3.1.12]. We conclude that  $C_0 = D$ . This clearly implies (ii). ■

**Corollary 1.4** *Any simple right  $C$ -comodule is isomorphic to  $S_x$  for some  $x \in X$ .*

**Corollary 1.5** *The set of all  $e_{x,y}$ 's with  $x \leq y$  is a coradical basis of  $C$ .*

For any  $x \leq y$  in  $X$  we consider the element  $p_{x,y} \in C^*$  such that  $p_{x,y}(e_{u,v}) = \delta_{x,u} \delta_{y,v}$  for any  $u, v \in X$  with  $u \leq v$ . If we denote by  $\rightarrow$  the left  $C^*$ -action on  $C$  induced by the right  $C$ -comodule structure, it is easy to check that

$$p_{x,y} \rightarrow e_{u,v} = \begin{cases} e_{u,x}, & \text{if } y = v \text{ and } u \leq x \\ 0, & \text{otherwise} \end{cases}$$

**Proposition 1.6** *For any  $x \in X$ ,  $S_x$  is an essential subcomodule of the right  $C$ -comodule  $E_x$ , thus  $E_x$  is the injective envelope of  $S_x$  in the category of right  $C$ -comodules.*

**Proof:** Let  $z = \sum_{x \leq y} \alpha_y e_{x,y} \in E_x$ ,  $z \neq 0$ . Choose some  $y_0$  such that  $\alpha_{y_0} \neq 0$ . Then  $p_{x,y_0} \rightarrow z = \alpha_{y_0} e_{x,x} \in S_x \setminus \{0\}$ . ■

**Corollary 1.7** *Any indecomposable injective right  $C$ -comodule is isomorphic to  $E_x$  for some  $x \in X$ .*

It is easy to describe the left  $C^*$ -submodule generated by some  $e_{u,v}$ .

**Proposition 1.8** *Let  $u \leq v$  in  $X$ . Then  $C^* \rightarrow e_{u,v} = \langle e_{u,z} \mid u \leq z \leq v \rangle$ .*

**Proof:** The inclusion  $\subseteq$  is clear since  $c^* \rightarrow e_{u,v} = \sum_{u \leq z \leq v} c^*(e_{z,v})e_{u,z}$ . On the other hand for any  $u \leq z \leq v$  one has  $p_{z,v} \rightarrow e_{u,v} = e_{u,z}$ , so each such  $e_{u,z}$  lies in  $C^* \rightarrow e_{u,v}$ , and the other inclusion is clear.  $\blacksquare$

In a similar way, the subspace  $S'_x = \sum_{y \in C_i} k e_{y,x}$  is a simple left  $C$ -subcomodule and  $E'_x = \langle e_{y,x} \mid y \in X, y \leq x \rangle$  is its injective envelope as a left  $C$ -comodule. Any indecomposable injective left  $C$ -comodule is isomorphic to some  $E'_x$ . There is a good left-right connection between simple comodules, the next result shows. We note that the left  $C^*$ -module structure of  $S_v$  induces a right  $C^*$ -module structure on  $S_v^*$ ; we denote this right action by  $\leftarrow$ . The usual right  $C^*$ -action on  $C$  is denoted by  $\leftarrow$ , too.

**Proposition 1.9** *If  $v \in X$ , then  $S_v^* \simeq S'_v$  as right  $C^*$ -modules.*

**Proof:** We denote by  $(\tilde{e}_{v,z})_{z \sim v}$  the basis of  $S_v^*$  dual to  $(e_{v,z})_{z \sim v}$ . Let  $\gamma : S_v^* \rightarrow S'_v$  be the linear isomorphism such that  $\gamma(\tilde{e}_{v,z}) = e_{z,v}$  for any  $z \sim v$ . We show that  $\gamma$  is a morphism of right  $C^*$ -modules. Indeed, if  $c^* \in C^*$ , then

$$\begin{aligned} \tilde{e}_{v,z} \leftarrow c^* &= \sum_{y \sim v} (\tilde{e}_{v,z} \leftarrow c^*)(e_{v,y}) \tilde{e}_{v,y} \\ &= \sum_{y \sim v} \tilde{e}_{v,z} (c^* \rightarrow e_{v,y}) \tilde{e}_{v,y} \\ &= \sum_{y \sim v} \sum_{u \sim v} c^*(e_{u,y}) \tilde{e}_{v,z}(e_{v,u}) \tilde{e}_{v,y} \\ &= \sum_{y \sim v} c^*(e_{z,y}) \tilde{e}_{v,y} \end{aligned}$$

so  $\gamma(\tilde{e}_{v,z} \leftarrow c^*) = \sum_{y \sim v} c^*(e_{z,y}) e_{y,v} = e_{z,v} \leftarrow c^* = \gamma(\tilde{e}_{v,z}) \leftarrow c^*$ .  $\blacksquare$

## 2 The main result

We recall that a coalgebra  $C$  is called right quasi-co-Frobenius if  $C$  can be embedded as a right  $C^*$ -module in a free right  $C^*$ -module. It is clear that any right co-Frobenius coalgebra is right quasi-co-Frobenius. It is known that a cosemisimple coalgebra is right co-Frobenius.

If  $(X, \leq)$  is a locally finite preordered set, then the factor set  $\tilde{X} = X / \sim$  with respect to the equivalence relation  $\sim$ , associated with  $\leq$ , is a locally finite partially ordered set (and there is no danger of confusion if we also denote the induced order relation on  $\tilde{X}$  by  $\leq$ ), so we can consider the corresponding incidence coalgebra  $IC(\tilde{X})$ . Now we can prove our main result. In the particular case where  $X$  is a partially ordered set (so then  $\tilde{X} = X$ ), we obtain [8, Corollary 3.2]. We note that even in this particular case, our approach is different from the one in [8].

**Theorem 2.1** *Let  $C = IC(X)$  be the incidence coalgebra of a locally finite preordered set  $X$ . The following assertions are equivalent.*

- (1)  *$C$  is right co-Frobenius.*
- (2)  *$C$  is right quasi-co-Frobenius.*
- (3)  *$C$  is cosemisimple.*
- (4) *For any  $x, y \in X$  such that  $x \leq y$ , we also have that  $y \leq x$  (or equivalently, elements in different equivalence classes with respect to  $\sim$  are not comparable).*
- (5)  *$IC(\tilde{X})$  is right co-Frobenius.*
- (6) *The order relation on  $\tilde{X}$  is the equality.*

**Proof:** (1)  $\Rightarrow$  (2) is clear.

(2)  $\Rightarrow$  (3) Any indecomposable injective right  $C$ -comodule is isomorphic to  $E_x$  for some  $x \in X$ . Take one such  $E_x$ . Since  $C$  is right quasi-co-Frobenius,  $E_x$  is isomorphic to the dual of an indecomposable injective left  $C$ -comodule  $P$  (see [11, Proposition 1.4] or [12, Proposition 1.3]), so it is local (see [10, Lemma 1.4]), and then by Proposition 1.2 it is generated as a left  $C^*$ -module by an element  $e_{u,v}$ . Thus  $E_x = C^* \rightarrow e_{u,v}$ .

Then  $e_{u,v} \in E_x$ , so necessarily  $u = x$ . Indeed, if  $u \neq x$ , then  $e_{u,v}$  cannot lie in the span of all  $e_{x,y}$ , with  $x \leq y$ . Thus we have  $E_x = \langle e_{x,z} \mid x \leq z \leq v \rangle$ , so  $\{y \in X \mid x \leq y\} = \{z \mid x \leq z \leq v\}$ .

Let  $\phi : E_x \rightarrow S_v$  be the linear map such that  $\phi(e_{x,s}) = e_{v,s}$  for any  $s \sim v$ , and  $\phi(e_{x,s}) = 0$  for any  $x \leq s < v$ . It is clear that  $\phi$  is surjective. We show that  $\phi$  is a morphism of left  $C^*$ -modules. Indeed, if  $s \sim v$  and  $c^* \in C^*$ , then

$$\begin{aligned} \phi(c^* \rightarrow e_{x,s}) &= \sum_{x \leq z \leq s} c^*(e_{z,s} \phi(e_{x,z})) \\ &= \sum_{z \sim v} c^*(e_{z,s}) e_{v,z} \\ &= c^* \rightarrow e_{v,s} \\ &= c^* \rightarrow \phi(e_{x,s}), \end{aligned}$$

while if  $x \leq s < v$ , then  $\phi(c^* \rightarrow e_{x,s}) = 0$  since  $z < v$  for any  $x \leq z \leq s$ , and  $c^* \rightarrow \phi(e_{x,s}) = c^* \rightarrow 0 = 0$ .

In a similar way to what we have done with right  $C$ -comodules, when we work with left  $C$ -comodules, an indecomposable injective left  $C$ -comodule is isomorphic to a comodule of the form  $E'_w = \langle e_{y,w} \mid y \in X, y \leq w \rangle$ , the indecomposable injective left  $C$ -subcomodule of  $C$  which contains  $e_{w,w}$ . Since  $C$  is right quasi-co-Frobenius, it must be right semiperfect (see [7, Corollary 3.3.6]), so  $E'_w$  is finite dimensional (see [7, Theorem 3.2.3]). Then we have a surjective morphism of left  $C^*$ -modules  $\theta : (E'_w)^* \rightarrow S_v$ , and this induces an injective morphism  $\psi : S_v^* \rightarrow E'_w$  of right  $C^*$ -modules. Since the socle of  $E'_w$  is just  $S'_w$ , we must have  $w = v$ , so then  $E_x \simeq (E'_v)^*$ .

Thus we have showed that if  $E_x = C^* \rightarrow e_{u,v}$ , then  $u = x$ ,  $\{y \mid x \leq y\} = \{z \mid x \leq z \leq v\}$ , and  $E_x \simeq (E'_v)^*$ .

Now since  $\{y \mid x \leq y\} = \{z \mid x \leq z \leq v\}$ , we see that if  $v \leq z$ , then  $z \sim v$ . It follows that  $E_v = S_v = C^* \rightarrow e_{v,v}$ . By the previous considerations for  $v$  instead of  $x$ , we obtain that  $E_v \simeq (E'_v)^*$ . It follows that  $E_x \simeq E_v$ , so then  $v = x$ . We conclude that  $E_x = S_x$ , and then

$C = \oplus_{x \in X} E_x = \oplus_{x \in X} S_x$  is cosemisimple.

(3)  $\Rightarrow$  (1) holds for any coalgebra.

(3)  $\Leftrightarrow$  (4) follows from the fact that  $S_x$  is the socle of  $E_x$ , so  $C$  is cosemisimple if and only if  $E_x = S_x$  for any  $x \in X$ .

(4)  $\Leftrightarrow$  (6) is clear.

(5)  $\Leftrightarrow$  (6) follows from (1)  $\Leftrightarrow$  (4) applied to  $(\tilde{X}, \leq)$ . ■

Since the cosemisimple property on coalgebras is left-right symmetric (or alternatively since the conditions (4) and (6) do not depend on the side we work on), the assertions of Theorem 2.1 are also equivalent to the left hand versions of (1), (2) and (5).

Let  $n$  be a positive integer and denote by  $e_{i,j}$  the matrix units in the matrix algebra  $M_n(k)$ . Let  $\mathcal{B} \subseteq \{1, \dots, n\} \times \{1, \dots, n\}$  be such that  $(i, i) \in \mathcal{B}$  for any  $i \in \{1, \dots, n\}$ , and  $(i, k) \in \mathcal{B}$  whenever  $(i, j) \in \mathcal{B}$  and  $(j, k) \in \mathcal{B}$ . Thus  $\mathcal{B}$  is a preorder relation on  $\{1, \dots, n\}$ . Then  $\mathcal{M}(\mathcal{B}, k) = \sum_{(i,j) \in \mathcal{B}} k e_{i,j}$  is a subalgebra of  $M_n(k)$ , called the structural matrix algebra associated to  $\mathcal{B}$ . It consists of all matrices whose  $(i, j)$ -entries are zero for  $(i, j) \notin \mathcal{B}$ . The dual coalgebra of  $\mathcal{M}(\mathcal{B}, k)$  is just the incidence coalgebra of the preordered set  $\{1, \dots, n\}$ , with the relation defined by  $\mathcal{B}$ . Since a finite dimensional algebra is Frobenius if and only if its dual coalgebra is right co-Frobenius (and then by the finite dimensionality it is also left co-Frobenius), we obtain as a consequence of Theorem 2.1 the following.

**Corollary 2.2** *A structural matrix algebra  $\mathcal{M}(\mathcal{B}, k)$  is Frobenius if and only if  $\mathcal{B} = (I_1 \times I_1) \cup \dots \cup (I_r \times I_r)$  for some partition  $I_1, \dots, I_r$  of  $\{1, \dots, n\}$ . In this case  $\mathcal{M}(\mathcal{B}, k) \simeq M_{n_1}(k) \times \dots \times M_{n_r}(k)$ , where  $n_1 = |I_1|, \dots, n_r = |I_r|$ .*

**Remark 2.3** *It is possible to prove Corollary 2.2 directly, using only ring theory methods. The referee indicated us one such proof. A structural matrix algebra may be put (up to an isomorphism) into upper block triangular form, see for instance [1, page 28]. As a consequence of [3, Chapter III, Proposition 2.7], the global dimension of a structural matrix algebra is finite. On the other hand, the global dimension of a Frobenius algebra is either 0 or  $\infty$  by [9, Theorem 11] or [2, Proposition 15]. Then if a structural matrix algebra is Frobenius, it has global dimension 0, so it must be semisimple, and then it consists only of diagonal blocks.*

We end by presenting a categorical connection between the two incidence coalgebras,  $C = IC(X)$  and  $IC(\tilde{X})$ . More precisely, we show that these two coalgebras are Morita-Takeuchi equivalent, i.e. their categories of right comodules are equivalent. Let  $\mathcal{S}$  be a system of representatives for the equivalence classes of  $X$  with respect to  $\sim$ , and let  $m \in C^*$  be such that  $m(e_{u,v}) = 1$  if  $u = v \in \mathcal{S}$ , and  $m(e_{u,v}) = 0$  for any other  $u \leq v$ . Thus  $m$  is just  $\varepsilon$  on  $E_u$  with  $u \in \mathcal{S}$ , and  $m$  is zero on any other  $E_u$ . Then  $m$  is an idempotent of  $C^*$ , and by [16, Lemma 6] we have that  $m \rightarrow C \leftarrow m$  has a coalgebra structure with the comultiplication defined by

$$\Delta'(m \rightarrow c \leftarrow m) = \sum (m \rightarrow c_1 \leftarrow m) \otimes (m \rightarrow c_2 \leftarrow m)$$

and counit just the restriction of  $\varepsilon$  to  $m \rightarrow C \leftarrow m$ . Note that  $m \rightarrow C \leftarrow m$  is not a subcoalgebra, but a factor coalgebra of  $C$ . Using the concept of a basic coalgebra defined in [5], and the terminology and results of [6, Section 3], we have that  $m$  is a basic idempotent of  $C$  and  $m \rightarrow C \leftarrow m$  is

the basic coalgebra of  $C$ , so then  $m \rightarrow C \leftarrow m$  is Morita-Takeuchi equivalent to  $C$  see [5, Corollary 2.2] or [6, Proposition 3.6].

By the way  $m$  is defined, it is easy to see that  $m \rightarrow e_{u,v} \leftarrow m = e_{u,v}$  if  $u, v \in \mathcal{S}$  and  $u \leq v$ , and  $m \rightarrow e_{u,v} \leftarrow m = 0$  in any other case. Thus  $m \rightarrow C \leftarrow m = \langle e_{u,v} \mid u, v \in \mathcal{S}, u \leq v \rangle$ , and its comultiplication works as

$$\begin{aligned} \Delta'(e_{u,v}) &= \sum_{u \leq y \leq v} (m \rightarrow e_{u,y} \leftarrow m) \otimes (m \rightarrow e_{y,v} \leftarrow m) \\ &= \sum_{u \leq y \leq v, y \in \mathcal{S}} e_{u,y} \otimes e_{y,v} \end{aligned}$$

This shows that the linear map  $f : (m \rightarrow C \leftarrow m) \rightarrow IC(\tilde{X})$  defined by  $f(e_{u,v}) = e_{\bar{u}, \bar{v}}$  for any  $u, v \in \mathcal{S}$  with  $u \leq v$ , is an isomorphism of coalgebras. Here we denoted by  $\bar{u}$  the class of  $u$  in the factor set  $X/\sim$ . We conclude that  $IC(X)$  and  $IC(\tilde{X})$  are Morita-Tacheuchi equivalent.

As a consequence, we obtain by duality that for a finite preordered set  $X$ , the incidence algebra of  $X$  is Morita equivalent to the incidence algebra of the partially ordered set  $\tilde{X}$ . Thus a structural matrix algebra, which is isomorphic to a blocked matrix algebra via a permutation of rows and columns, is Morita equivalent to a more simple structural matrix algebra, which is presented in the block form with blocks of size 1.

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